THE
MATHEMATICAL GAZETTE.

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NORTH WALES BRANCH.

A MEETING of this branch was held on February 20th at the Friars' School, Bangor—the President, Dr. Bryan, in the chair. Mr. R. W. Jones, Headmaster of Glanadda Elementary School, opened a discussion on the teaching of the foundations of arithmetic to children of 6 or 8 years old. After referring to the difficulty of forming any ideas as to a child's conception of number, and mentioning instances of the vague answers given by a child who is asked to estimate some large number of objects, Mr. Jones proceeded to show how the four fundamental laws of arithmetic could be gradually and insensibly taught to children by means of experiments involving simple operations in weighing and measuring. He emphasised the necessity for using familiar illustrations in expounding the four rules and brought out two points as worthy of special notice: (1) that in teaching the elements to children it is absolutely necessary to confine one's attention to *small* numbers; (2) that the time spent in teaching the multiplication tables to children of 6 or 8 years was not well spent, since for small numbers at any rate, they could be learnt insensibly, and the higher tables could be learnt with much greater facility in the child's third year at school. Most of those present took part in the interesting discussion which followed, generally expressing their agreement with Mr. Jones's views.

Mr. H. F. D. Turner, the mathematical master at the Friar's School, then introduced the subject of entrance scholarship examinations at the Welsh University Colleges, urging the necessity for sympathetic and concerted action between the school and University authorities and the Central Welsh Board. He considered the case of a boy on the point of leaving

school at 18 years of age, who has specialised in mathematics and physics during his last two years. On deciding to go to the University he finds that he has four subjects in his entrance scholarship examination, thus giving him two extra subjects which he has most probably to prepare hurriedly to the detriment of his mathematics and physics. On the other hand a less brilliant candidate who has shown no tendency towards specialisation might easily keep up his four subjects to such a standard as would enable him to score an aggregate of marks equal to that of the more highly specialised candidate. Further, in the advanced scholarship papers the questions set are not of a very high standard, requiring a good general knowledge of somewhat elementary work rather than any specific knowledge of the higher branches of study.

Several members criticised the tendency towards specialisation in schools, and Dr. Bryan pointed out, with regard to the scholarship examinations, that nowadays all students on entering the University were supposed to have matriculated. This was not the case in earlier times, and the scholarship papers then set were supposed to be of matriculation standard, a standard which has since been more or less roughly adhered to.

It was decided that the next meeting should be held on May 22nd at Beaumaris.

THE DIVISION OF THE CIRCLE—*Concluded.*

To establish the order of the locus we must note that besides the above mentioned n intersections with the circumference, the point D also belongs to the curve, so that the latter cuts the circle in $n+1$ points. The order of the curve is therefore $\frac{n+1}{2}$.

Let n be even.

The ratio $k=\frac{n-1}{2}=p-\frac{1}{2}$ is not an integer, so that a complete rotation of the straight line BC must take place before the two generating straight lines can be in their initial mutual position; the line revolving round D will be, after a semi-rotation of BC, parallel to BC in DE, and we may say that all the loci for $n=2p$ are asymptotic in the direction BC, on account of the initial position of BC and DA.

The point M', the image of M in B, is a point on the locus.

As n is an even number this must be so, but the proof follows easily from the equality of the angles $M'DE=ADM$, whose sides are perpendicular to one another, two by two; thus

$$\frac{M'BC}{EBM'}=\frac{ADM}{MBC}=\frac{n-1}{2}. \qquad \text{Q.E.D.}$$

Also M_2, making $MM_2 = \frac{\pi}{n}$, is a point on the locus. If BM' rotate through the arc $\frac{\pi}{n}$, DM' should cross in M_2, having rotated through the arc $\frac{n-1}{2} \cdot \frac{\pi}{n}$. In fact

$$M'DM_2 = \frac{1}{2}\left(\pi - \frac{\pi}{n}\right) = \frac{n-1}{2}\,\frac{\pi}{n}.$$

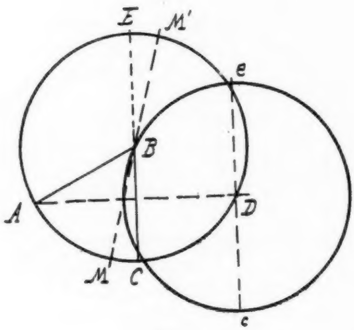


FIG. 8.

So also for n even, the vertices of the regular polygon, which are the intersections of the locus with the circle, are $2n$.

The point D is a double point of the curve, since it passes through D at every semi-rotation of the line in a different direction. It follows that there are $2n+2$ intersections with the circle, and the order of the curve is $n+1$.

A more general remark applies to the order of these n-secting curves, and will come in useful for a result that will be given later.

Let us imagine two straight lines rotating one round A, the other round B, with the respective velocities α and β; let α and β have no common factor: what will be the order of the locus of intersections of the corresponding straight lines?

Let α be $< \beta$; after α turns of the straight line rotating round A it will have passed α times through B, and the straight line rotating round B will have passed β times through A, and the two lines returning to their initial mutual position will have generated the whole locus; this curve will then have β tangents at A and α tangents at B, and will not be cut in any other points by the straight line AB unless AB itself form part of the curve. The order of the curve will then be $\alpha + \beta$.

As verification we note that the ratio reduces to the form $\frac{n-1}{2}\Big/1$; the order of the locus according to the above rule is $\frac{n-1}{2}+1=\frac{n+1}{2}$, as already found.

For n even, the ratio $\frac{n-1}{2}$ is not susceptible of further reduction, and the order of the curve is therefore $n-1+2=n+1$, as already found.

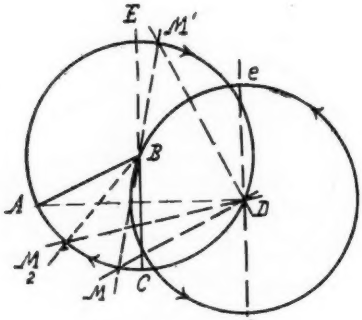


FIG. 4.

Can the problem of n-section of the angle on the circle be solved by curves of other orders with different values of K?

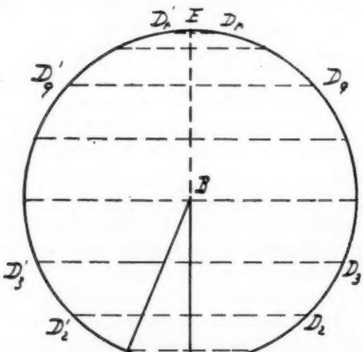


FIG. 5.

Let D_1, D_2, D_3 ... D_p on the circumference be placed at equal distances $CD=D_1D_2=D_2D_3=\ldots AC$.

Let B remain the centre of rotation of BC, and let D_2, $D_3 \dots$ be successively centres of rotation for the straight line DA. To obtain the intersection of each locus in M, such that $\frac{AC}{MC}=n$, we must use the following different ratios of velocity:

$$\begin{array}{lll} \text{Centre } & D_1, & k=\frac{n-1}{2}, \\ \quad " & D_2, & k=\frac{2n-1}{2}, \\ \quad " & D_3, & k=\frac{3n-1}{2}, \\ \dots\dots & \dots\dots & \dots\dots \\ \quad " & D_p, & k=\frac{pn-1}{2}. \end{array}$$

D_p is the last centre of the series which is not beyond E.

The order of all the curves corresponding to the above ratios can be deduced from the same values of k under the form of fractions in their lowest terms.

When the points A, D'_2, $D'_3 \dots D'_p$ the images of D_1, D_2, $D_3 \dots D_p$ in the diameter EC are chosen as centres of rotation and the sense of rotation be reversed but still keeping the initial position of the straight line perpendicular to BC, then the other curves are obtained involving the following ratios of velocity:

$$\begin{array}{lll} \text{Centre at } & A, & k=\frac{n+1}{2}, \\ \quad " & D'_2, & k=\frac{2n+1}{2}, \\ \quad " & D'_3, & k=\frac{3n+1}{2}, \\ \dots\dots & \dots\dots & \dots\dots \\ \quad " & D'_p, & k=\frac{pn+1}{2}. \end{array}$$

Transferring the centre of revolution from B to E, and keeping the same initial position for the straight line with respect to BC, another family of the curves is obtained, corresponding, however, to values of k double of those given above, *i.e.*

$$k=\frac{pn-1}{1}, \quad k=\frac{pn+1}{1},$$

according as the movement of the straight lines be in the same or in opposite senses.

Pairs of points, images in respect of EC, such as A and D_1, D_2 and $D'_2 \dots D_p$ D'_p, may be taken as centres of revolution. The family of curves thus generated involves the following values of

k; centres in A and D, corresponding straight lines coincident in the initial position:

$$k=\frac{n+1}{n-1}.$$

Centres in D_2 and D'_2, straight lines as before,

$$k=\frac{2n+1}{2n-1}.$$

Centres in D_p and D'_p, straight lines as before,

$$k=\frac{pn+1}{pn-1}.$$

Assuming as centres two such points as D_p and D_q, then must $k=\frac{pn+1}{qn-1}$ (the senses of revolution being opposite and the initial direction of the straight lines, parallel to one another, being perpendicular to EC).

Or for two centres such as D'_p and D'_q, we have

$$k=\frac{pn+1}{qn+1} \quad \text{(rotation in same sense).}$$

And for two centres such as D_p and D_q

must
$$k=\frac{pn-1}{qn-1} \quad \text{(rotation in same sense).}$$

Such are the possible values of k giving n-secting curves for all combinations of centres chosen among the series of points mentioned; but k must be a definite function of n. For particular positions of the straight lines the values of k may be expressed by a constant independent of n; the circle itself is the locus defined by any such values of k as are useless.

Genoa, 1907. CAMILLO MANZITTI.

NOTE ON FOURIER'S THEOREM.

PROFESSOR BRYAN's paradox on Fourier's theorem (*Gazette*, vol. iv. p. 390) seems to call for a few words of explanation.

In the first place a function $f(\theta)$ is constructed of the form

$$f(\theta)=A_0+A_1\cos\theta+A_2\cos 2\theta+\dots+A_{n-1}\cos(n-1)\theta$$
$$+B_1\sin\theta+B_2\sin 2\theta+\dots+B_{n-1}\sin(n-1)\theta,$$

and the function is arranged so as to have the same values as an arbitrary function $F(\theta)$ at the places $\theta=0,\ a,\ 2a,\ \dots,\ (n-1)a$, where $a=2\pi/n$. It will be observed here that we have only n *data* from which to determine $(2n-1)$ coefficients, and consequently the form of $f(\theta)$ is largely at our disposal; but the particular form selected by Professor Bryan leads to the formulae for the coefficients,

$$nA_r=\sum_{s=0}^{n-1}F(sa)\cos rsa,\quad nB_r=\sum_{s=0}^{n-1}F(sa)\sin rsa.$$

The next step in the argument is to make n tend to ∞; it is evident, from the definition of a definite integral, that

$$\lim A_r = \frac{1}{2\pi}\int_0^{2\pi} F(\theta)\cos r\theta\, d\theta, \quad \lim B_r = \frac{1}{2n}\int_0^{2\pi} F(\theta)\sin r\theta\, d\theta,$$

or
$$\lim A_r = \tfrac{1}{2}a_r, \quad \lim B_r = \tfrac{1}{2}b_r \quad (r=1, 2, 3 \ldots),$$

where a_r, b_r are the ordinary Fourier constants of $F(\theta)$. On the other hand, $\lim A_0 = a_0$, and so we *seem* to get

$$\lim f(\theta) = a_0 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r\cos r\theta + b_r\sin r\theta),$$

whereas we might hope to get Fourier's theorem,

$$F(\theta) = a_0 + \sum_{r=1}^{\infty}(a_r\cos r\theta + b_r\sin r\theta).$$

As a matter of fact, however, there is no reason to suppose that $\lim f(\theta)$ is equal to $F(\theta)$; to see this let us take the simplest case of all, namely $F(\theta) = \cos\theta$. The formulae will then be found to give

$$A_0 = 0,\ A_1 = \tfrac{1}{2},\ A_2 = 0,\ \ldots,\ A_{n-2} = 0,\ A_{n-1} = \tfrac{1}{2},$$

while all the B's are zero. Hence

$$f(\theta) = \tfrac{1}{2}\{\cos\theta + \cos(n-1)\theta\},$$

which will be seen at once to be equal to $\cos\theta$ at the places

$$\theta = 0,\ \alpha,\ 2\alpha,\ \ldots,\ (n-1)\alpha.$$

But $f(\theta)$ does *not* tend to $\cos\theta$ as a limit, because $\cos(n-1)\theta$ has no definite limiting value (except for special values of θ), as n tends to ∞.

Returning now to the general formula, there is another step in the limiting process which calls for remark; it has, in fact, been tacitly assumed that if v_r is a function of r, n,* then

$$(1) \qquad \lim_{n\to\infty}\sum_{r=0}^{n-1} v_r = \sum_{r=0}^{\infty} w_r, \quad \text{where } w_r = \lim_{n\to\infty} v_r.$$

This equation (1), however, need not be true: a simple example is given by taking

$$v_r = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{(r+1)^2}, \text{ so that } w_r = \frac{C}{(r+1)^2}.$$

Then
$$\sum_{r=0}^{n-1} v_r = A + \frac{B}{n} + \sum_{r=0}^{n-1}\frac{C}{(r+1)^2},$$

so that
$$\lim_{n\to\infty}\sum_{r=0}^{n-1} v_r = A + \sum_{r=0}^{\infty}\frac{C}{(r+1)^2} = A + \sum_{r=0}^{\infty} w_r,$$

and consequently equation (1) is true only if $A=0$.

The equation (1) occurs frequently in various elementary calculations, such as the exponential limit; and on this account it may be worth while to call attention to a simple test for its validity, which appears to be due to Tannery. *The equation* (1) *will be true if we can find a convergent series of positive constants* ΣM_r, *such that* $M_r \geqq |v_r|$, *for all values of* n; for a proof I refer to Art. 49 of my book on *Infinite Series*.

Closely allied with Professor Bryan's paradox is a discussion (due to Lagrange) which will be found in Todhunter's *Integral Calculus* (Art. 306) or in Byerly's *Fourier's Series*, etc. (Arts. 19–23); here the function $f(\theta)$ is taken to be

$$f(\theta) = C_1\sin\theta + C_2\sin 2\theta + \ldots + C_{n-1}\sin(n-1)\theta,$$

* In the case considered above v_r is $A_r\cos r\theta + B_r\sin r\theta$.

and the coefficients $C_1, C_2, \dots C_{n-1}$ are determined by making $f(\theta)=F(\theta)$ at $\theta=a, 2a, \dots (n-1)a$, where $a=\pi/n$. It is then found that

$$nC_r=2\sum_{s=1}^{n-1}F(sa)\sin rsa,$$

and so $$\lim_{n\to\infty} C_r=\frac{2}{\pi}\int_0^\pi F(\theta)\sin r\theta\, d\theta=c_r, \text{ say.}$$

Then the inference is that

$$F(\theta)=\lim_{n\to\infty}f(\theta)=c_1\sin\theta+c_2\sin 2\theta+\dots,$$

and actually the series so obtained *is* the Fourier sine-series for $F(\theta)$. But from what has been said already, it will be clear that the equation

$$(2) \qquad F(\theta)=\lim_{n\to\infty}f(\theta)$$

is far from evident; and in addition the Tannery test for equation (1) will not justify its validity here. Before leaving this question it is perhaps worth while to call attention to the fact that *no investigation of Fourier's theorem can be correct which does not explicitly introduce some restriction on the character of the function* $F(\theta)$. For continuous functions have been constructed so that the corresponding Fourier series do not converge but oscillate.

Thus any treatment on the lines indicated above, even when it leads to an apparently correct result (as in the second case), must be regarded only as a rough approximation to a proof. T. J. I'a Bromwich.

I am greatly indebted to Prof. Bromwich for his explanation of my difficulty. The mistake is one that might easily be made by a teacher of physics or engineering, and affords a fitting illustration of the sort of thing that may happen if the mathematics of this country is all reduced to the B.Sc. standard, and the specialist starved out of existence instead of being called in to explain difficulties. It appears desirable, however, to pick up the fragments of the exploded proof and try and cement them together.

The fallacy, as Prof. Bromwich points out, consisted in trying to determine $2n-1$ constants from n data. The following considerations clear up the difficulty.

(1) We notice that at each of the points on the circle

$$\cos r\theta=\cos(n-r)\theta, \text{ and } \sin r\theta=-\sin(n-r)\theta. \qquad \text{(1)}$$

The value of the series at each of these points will therefore be unaltered by adding arbitrary terms of the form

$$\Sigma C_r(\cos r\theta-\cos(n-r)\theta)+D_r(\sin r\theta+\sin(n-r)\theta).$$

In particular if n is odd and equal to $2m+1$ we may use the above relations to replace multiple angles greater than $m\theta$ by multiples less than $m\theta$. This replaces the series by a new series of sines and cosines of multiples of angles up to $m\theta$ only.

(2) In multiplying both sides of the equation

$$F(\theta)=\Sigma A_r\cos r\theta+B_r\sin r\theta,$$

and taking the mean values for the n points round the circle, the fact was overlooked that the mean values of $\cos r\theta\cos(n-r)\theta$ and $\sin r\theta\sin(n-r)\theta$ do not vanish. The former is in fact equal to $\frac{1}{2}$, and the latter is equal to $-\frac{1}{2}$. The corrected equation then stands,

$$\left.\begin{aligned}\tfrac{1}{2}(A_r+A_{n-r})&=\text{mean value of } F(\theta)\cos r\theta,\\ \tfrac{1}{2}(B_r-B_{n-r})&=\text{mean value of } F(\theta)\sin r\theta\end{aligned}\right\}. \qquad \text{(2)}$$

(3) In any series it is easily seen that

$$A_{n-r}=A_r \text{ and } B_{n-r}=-B_r,$$

whence by (1) the terms equidistant from the beginning and end are equal. It is obvious that if we make the number of terms infinite in such a case half the series goes off to infinity and gets cut off. It is thus shown that the only possibility of getting a series which can hold in the limiting case of $n\to\infty$ is by replacing the terms after the middle by terms before the middle, reducing the series to one of half the length as explained under (1) above. In illustration we may draw out a band of indiarubber as much as we like so long as the length remains finite, but if one end be drawn out to infinity we shall only have half the band left.

When these necessary corrections and modifications are made we do get a series which tends to Fourier's series as its limit.

G. H. BRYAN.

HOMOGRAPHIC RANGES—ELEMENTARY PRINCIPLES.

MY object is to indicate what seem to be the fundamental ideas connected with homography which would be readily understood and appreciated by a beginner.

Ranges in perspective.

If a pencil of concurrent rays is cut by any transversal, the ratios between the parts into which it is divided are the same as for any parallel transversal. This is not the case when the two transversals are not parallel, and it becomes an important problem to find out what property of the segments is common to both transversals in such a case. To the Greeks, who so exhaustively studied the properties of the conic sections as sections of a cone, this problem must have presented itself quite early, and it appears that not only Pappus, but also Euclid, and therefore probably Apollonius, knew the solution.

The modern way might be as follows:

Let any three rays OA, OB, OC cut the transversal in the points A, B, C; then

$$\frac{AC}{BC}=\frac{AC}{OC}\cdot\frac{OC}{BC}=\frac{\sin AOC}{\sin A}\cdot\frac{\sin B}{\sin BOC}=\frac{\sin AOC}{\sin BOC}:\frac{\sin A}{\sin B},$$

so that the value of the ratio depends on the angles at A and B as well as on those at O.

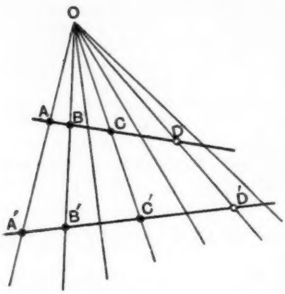
Now, let any 4th ray OD cut the transversal in D; then

$$\frac{AD}{BD}=\frac{\sin AOD}{\sin BOD}:\frac{\sin A}{\sin B},$$

so that both $\frac{AC}{BC}$ and $\frac{AD}{BD}$ depend *in the same way* upon the angles at A and B.

Hence their ratio, viz. $\frac{AC}{BC}:\frac{AD}{BD}$, is independent of everything except the angles at O.

Therefore this ratio of ratios, or *cross-ratio* as Clifford called it, is the same for all transversals cut by the four rays OA, OB, OC, OD.



Similarly for any number of rays: if any four of them are taken, all transversals are cut by them in equal cross-ratios.

The importance of this discovery can hardly be exaggerated: it at once laid all the conic sections at the feet of the circle and lines connected with it.

This equality of cross-ratios is the fundamental property of transversals cutting a pencil of concurrent lines, but the old geometers put this property in different ways, some of which are more convenient when considering the whole series of corresponding points on two transversals.

Thus, if we write the relation

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{A'C'}{B'C'} : \frac{A'D'}{B'D'}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

in the form
$$\frac{AD}{BD} = \left\{\frac{AC}{BC} : \frac{A'C'}{B'C'}\right\} \cdot \frac{A'D'}{B'D'},$$

$$\text{i.e. } \frac{AD}{BD} = \mu \frac{A'D'}{B'D'}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where μ is the constant quantity $\frac{AC}{BC} : \frac{A'C'}{B'C'}$,

we see how D moves on one transversal as D' moves on the other, both transversals being entirely filled with pairs of corresponding points, and we see also that the correspondence is entirely dependent on the six points ABC, $A'B'C'$.

Or, again, if a point I is found on the first transversal corresponding to ∞' on the second, and J' on the second corresponding to ∞ on the first, we obtain the relations

$$IA \,.\, J'A' = IB \,.\, J'B' = \text{etc.} = \lambda \text{ (say)}. \ldots\ldots\ldots\ldots\ldots(3)$$

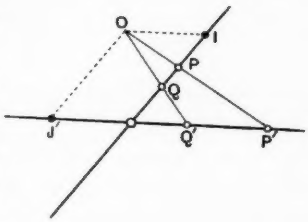
This is easily deduced algebraically by taking I at C, and J' at D', but the geometrical proof is simple and instructive.

Let P, P' be any pair of corresponding points, then

$$\frac{IP}{OI} = \frac{OJ'}{J'P'}.$$

$$\therefore\ IP \,.\, J'P' = IQ \,.\, J'Q' = \ldots.$$

When ranges of points on two straight lines satisfy these algebraic conditions the ranges are said to be homographic, and it should be noted that any one of these three conditions involves the others. It should be noted also that ranges which have these algebraic relations are still called homographic even when they are moved so as to be no longer in perspective.



Returning again to the four points $ABCD$: there are six different cross-ratios between the segments, depending on the way in which the segments are taken; and each of these is equal to the corresponding cross-ratio on any other transversal.

That there are six different cross-ratios is of course due to the fact that there are six simple ratios between the primary segments determined by the three points A, B, C, viz.

$$\frac{AC}{BC}, \frac{BC}{AC}; \frac{BA}{CA}, \frac{CA}{BA}; \frac{CB}{AB}, \frac{AB}{CB};$$

the corresponding cross-ratio being completed in each case by dividing by the correlative ratio in which D takes the place of the doubled letter, *i.e.*

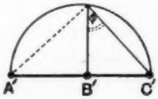
$$\frac{AD}{BD}, \frac{BD}{AD}; \frac{BD}{CD}, \frac{CD}{BD}; \frac{CD}{AD}, \frac{AD}{CD};$$

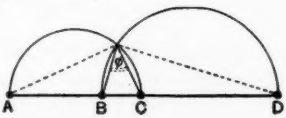
and that there are not more than six is evidenced by the fact that if a transversal is drawn parallel to OD, cutting the rays in A', B', C', each of the above cross-ratios equals one of the six simple ratios between the segments of this new transversal, in the same order as in the undashed letters.

Two of these are obviously negative, while the other four are positive.

They are equal to functions of an angle ϕ obtained by one or other of the two figures shown, overlapping semicircles being drawn on the segments.

The negative ratios are $-\tan^2\phi$, $-\cot^2\phi$.

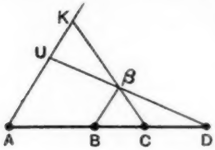




It would be good to exercise students in evaluating or forming cross-ratios of definite magnitude: the easiest method seems to be as follows:

Consider (AB, CD), *i.e.* $\dfrac{AC}{BC} : \dfrac{AD}{BD}$.

Draw parallels AK, $B\beta$ through A, B and on AK mark off AU = unity on a convenient scale. Join DU, cutting $B\beta$ in β.



Let $C\beta$ cut AU in K; then AK is the value of the cross-ratio.

(In particular this method gives a very neat construction for a 4th harmonic when three points are given.)

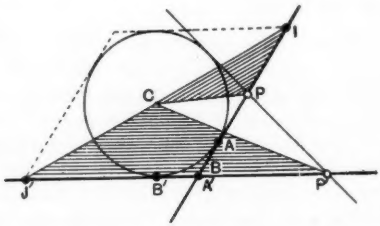
Coplanar homographic ranges in general.

The consideration of the converse of the theorem that concurrent lines determine homographic ranges on transversals must have also been of great interest to the old geometers.

The converse does not hold unless the transversals intersect in a pair of corresponding points, but when that condition is fulfilled it *is* true that the rays joining corresponding points are concurrent, and a most important theorem it is.

In the general case the rays are not concurrent, but envelop some conic section, as can readily be proved by means of the following theorem. The proof of the theorem will be based on the homographic relation $IA \,.\, J'A' = \text{constant}$.

To prove that a variable tangent to a circle cuts two fixed tangents homographically.



The figure explains itself. IP, $J'P'$ are the fixed tangents and PP' is the variable one, the points I, J' being opposite corners of a rhombus of tangents, the diagonal IJ' being bisected at the centre C of the circle. The shaded triangles are similar, as can be proved without much difficulty, so that

$$\frac{IP}{CI} = \frac{J'C}{J'P'},$$

whence $IP \,.\, J'P' = CI^2 = \text{constant}$;

$\therefore$ the ranges $(P, \ldots)$ and $(P', \ldots)$ are homographic.

By projection the theorem is also true of any conic, and now we are ready for the complete converse of the fundamental theorem relating to homographic ranges, viz. that the rays which join pairs of corresponding points of two homographic ranges meet in a point if the ranges intersect in a pair of

corresponding points, but in all other cases they envelop a conic which touches the lines containing the ranges. For, three of the rays determine the homography, and these with the two fixed lines can be touched by a conic, and then if *any* point on one range is joined to its corresponding point on the other, the joining line must touch the conic.

Of course the conic may be a point. This is the perspective case with which we started.

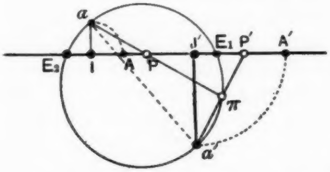
Co-axial homographic ranges—Involution.

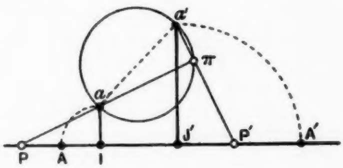
Lastly, if two homographic ranges are placed on the same line, we have common points to consider, where a pair of corresponding points coincide with each other.

The position of such points is given by the quadratic equation

$$IE\,.\,J'E = IA\,.\,J'A',$$

$$i.e.\quad IE^2 - IJ'\,.\,IE = IA\,.\,J'A'.$$





One construction is as follows:

Turn IA, $J'A'$ opposite ways through a right angle, as Ia, $J'a'$, and describe a circle on aa' as diameter. This circle will cut IJ' in the two possible positions of E.

A neat construction for any number of pairs of corresponding points on the axis is obtained by joining a, a' to any point π on this circle. If πa, $\pi a'$ cut the line IJ' in points P, P', these points P, P' will be a pair of corresponding points. [It is obvious, from similar triangles, that $IP\,.\,J'P' = Ia\,.\,J'a'$.] Two diagrams are given, one with real double points, and the other with imaginary double points. The construction of any number of pairs of corresponding points is equally simple in both diagrams.

The most important case of collinear homographic ranges is that in which I and J' coincide. The ranges are then said to be in involution, and in this case, and in this case only, any point of the line has the same conjugate whether the point selected belongs to one range or the other; *i.e.* in this case

$$IA\,.\,J'A' = IA'\,.\,J'A.$$

This peculiarity can happen only when I and J' coincide, for the above equation reduces to $IJ'\,.\,AA' = 0$. $\therefore$ unless I and J' coincide the points A, A' must, in which case they are one of

the common points E_1, E_2 of coaxial ranges which are not in involution (as in the previous section).

This is a brief outline of the main homographic theorems, and I have ventured to bring them before the meeting in this condensed form to show how these main features can be brought home to students forcibly and simply, before they begin to work through a detailed treatise. A. LODGE.

In response to a request from Mr. Lodge the Rev. John J. Milne gave the following historical sketch illustrative of the points touched upon in Mr. Lodge's paper:

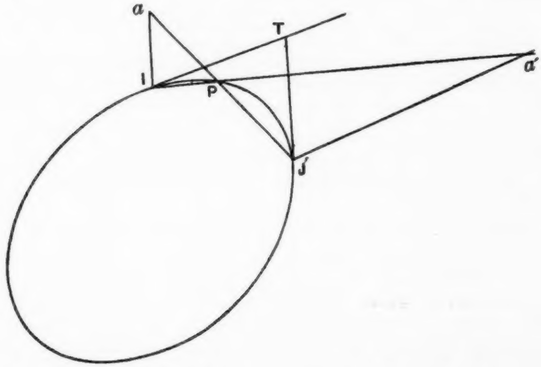
"The foundation-stone of the subject is the first property mentioned by Mr. Lodge, viz. the equality of the cross-ratios of all transversals cut by a pencil of four lines, which we first meet with in Pappus (300 A.D.). This does not mean that Pappus discovered it, for he merely gives it as one of about 30 Lemmas which he says would be helpful to a student in reading Euclid's Treatise on Porisms, so that the probability is that frequent use was made of it by Euclid (290 B.C.). Poncelet (1822) thought that the Porisms were a treatise on projective geometry, but Chasles (1860), who went very thoroughly into the question, was convinced that of the three books of the Porisms the first treated of the principles of homography of ranges on two separate lines, the second considered ranges on the same line, and the third treated of the anharmonic properties of points on a circle. This of course is to some extent a matter of conjecture, but I think that anyone who studies carefully the nature of the Lemmas given by Pappus, and his remarks respecting Porisms in the introduction to the seventh book of his Mathematical Collections will agree that the probability is that Chasles' view is correct, and that the theory of cross-ratios was well known to Euclid.

"The notation $O(abcd)$ is certainly modern, and was introduced by Möbius (1827). The extension of the principles of cross-ratios to more than four points is due to Chasles, who invented the term homography, corresponding points being linked together (ὁμοῦ), and developed the theory in his *Géométrie Supérieure* (1852), where he was not quite so happy as usual in his method of finding the common points of two coaxial ranges, as he obtains them by constructing five circles, which is quite impracticable. The usual method given in textbooks is by projection on to a conic or circle, but this really involves a knowledge of the anharmonic properties of a circle, and also of Pascal's hexagram, and therefore a teacher has here to leave the straight line, and take up the theory of the circle or conic and then return to the straight line, which seems not quite logical, and an interruption of the natural order. This can now be avoided by the method which Mr. Lodge has given us in his paper, and which was

discovered by him in 1907. It only requires a knowledge of the sixth book of Euclid, and it also enables us to construct a range homographic to a given coaxial range.

"The theory of Involution was developed by Desargues (1593–1662), who gave the seven equations which exist between the segments of six points in involution; but here again he was anticipated by Pappus, who gives us the relations between the segments on a transversal made by the opposite sides and diagonals of a quadrilateral.

"When we pass on to the conic we find four fundamental theorems. The first is 'Four fixed tangents cut any fifth in a constant cross-ratio.' This was discovered independently by Steiner and Chasles about 1830, and so was its correlative 'The pencil formed by joining four fixed points on a conic to any fifth point on it has a constant cross-ratio.' Then we come to the two most important, and most fascinating theorems in the whole range of conics, viz. 'Given two homographic ranges on two straight lines, the lines joining pairs of corresponding points envelop a conic,' and its correlative 'Given two homographic pencils, the intersections of corresponding rays lie on a conic.' These were first given by Chasles in his *Aperçu historique* (1837), and he was under the impression that he was their discoverer. But about a month ago I was looking through the Conics of Apollonius (250 B.C.) trying to find something else, and in that wonderful third book of his, most of the propositions of which, he tells us, were discovered by himself, I found this property.



"I, J' are two fixed points on a conic, Ia, $J'a'$ lines parallel to the tangents at J', I, and P any variable point on the curve. Then if $J'P$, IP meet these lines in a, a', $Ia \cdot J'a'$ is constant for all positions of P.

"Consequently, as Mr. Lodge has pointed out in his paper, the ranges (a) and (a') are homographic, and therefore so also are the pencils $I(a')$ and $J'(a)$, and Apollonius' proposition is neither more nor less than a statement of what Chasles called the anharmonic property of points on a conic, so that the germs of the theory, both as regards the line and conic, had their birth more than 2000 years ago. As a French writer somewhere remarks, 'It seems as though ideas resemble ourselves in having an infancy and period of feebleness. When they are first born they are unproductive, and it is only by age and time that they acquire their powers of fertility.'

"There is just one more remark that I should like to make before I sit down. Speaking as a schoolmaster to schoolmasters I think we ought to bring the history of mathematics more than we do before the notice of our pupils. Mathematics is a living, growing science, with a definite history, and there is not a branch of it which boys take up in school, whether Arithmetic, or Algebra, or Geometry, or Trigonometry or any other of its many divisions, but has its own history, and I have always found that boys are interested in learning what properties were known to the ancients, and what have been discovered in modern times, and I often think that the writers of our text-books would do well to devote a little more space than they do to what I may call 'the note of human interest.'"

MATHEMATICAL NOTES.

289. [K. 1.] *Note on Euclid I.* 16, 27.

These propositions are true in plane but not in spherical geometry.* Their truth therefore essentially depends on the axiom that two straight lines cannot enclose a space. Yet no reference to this axiom is made by Euclid or those who follow him in the matter of I. 16. It is the figure and not the logic which convinces in this case. A similar remark applies to the attempt, Note 230, vol. iv. p. 19, to prove I. 27, "assuming nothing but I. 4 and 13." The propositions explicitly assumed, and also axiom 8, which is tacitly used, are all true in spherical geometry. But the result to be proved is not true in spherical geometry. The simplest proof of I. 27 is that given by Halsted, *Rational Geometry*, § 66.

E. J. NANSON.

290. [D. a.] The sum of the series in 613 (2), No. 68, p. 171, should read

$$\frac{1}{2}\left[\frac{3^{n+2}}{n+3}+\frac{3^{n+1}}{n+2}-\frac{5}{2}\right].$$

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* "This proposition is not universally true under the Riemann hypothesis of a space endless in extent but not infinite in size." v. *The Thirteen Books of Euclid's Elements.* T. L. Heath. Vol. I. p. 280. And again, p. 309. "De Morgan observes that I. 27 is a logical equivalent to I. 16. Thus if A means 'straight lines forming a triangle with a transversal,' B, 'straight lines making angles with a transversal on the same side which are together less than two right angles,' we have:—all A is B, and it follows *logically* that all not $-B$ is not $-A$." [Ed.]

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